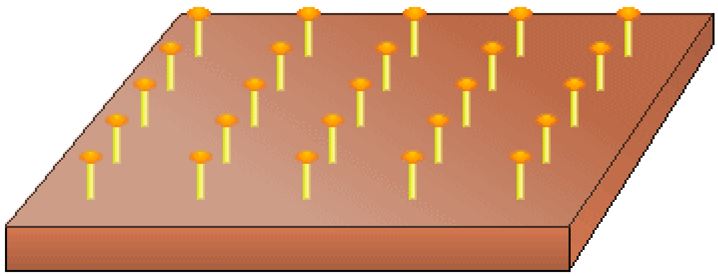
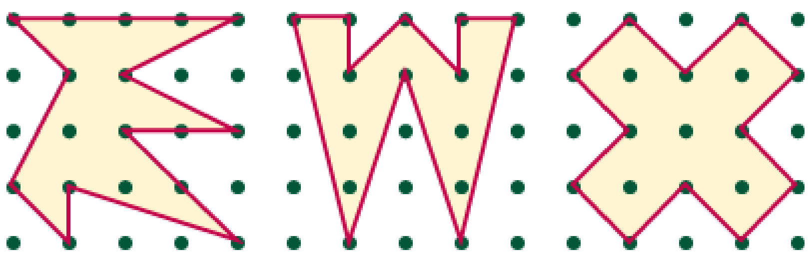
Activity 3
The Geoboard

The geoboard is a device used in elementary schools to aid in the teaching of basic geometric concepts. Geoboards may be purchased commercially from the usual supply houses or they may be constructed out of common household materials using common tools. A simple geoboard can be made from a square piece of wood and 25 finishing nails. A grid of 5 vertical lines and 5 horizontal lines evenly spaced are drawn on the square piece of wood. Nails are placed at the intersections of the lines so that they extend about one centimeter.



Figures are made on the geoboard by stretching rubber bands from one nail to another until the desired shape is formed. Segments can be shown by connecting only two nails. The first task to perform is to determine how many different segments may be constructed on the geoboard. As with most of the problem solving situations we will encounter, we will first simplify the problem, then look for patterns to help solve it, and then attempt to generalize the solution. It is easier to work with an actual geoboard than to work on paper alone, but if a geoboard is not available, dot paper can be used. The dot paper below shows some figures drawn as if on a geoboard.

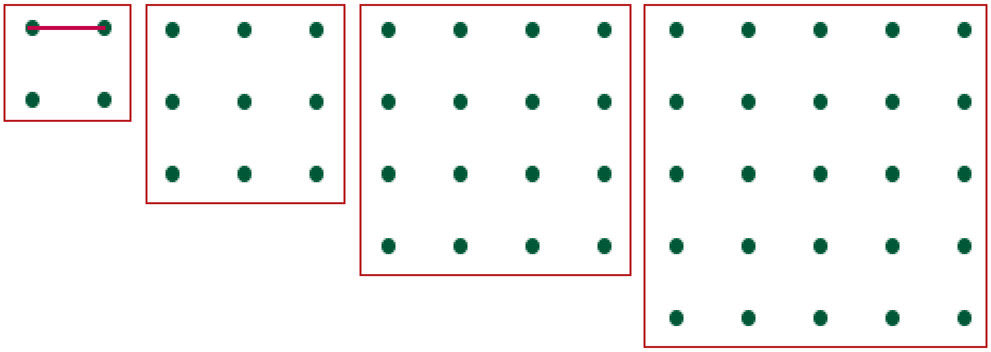


Segments on a Geoboard

Take a geoboard and some rubber bands. Assuming that the shortest distance between two adjacent nails is one unit, how many segments could be constructed with a length on one unit? Suppose the geoboard was a 4 by 4 nail version. Then how many? Find the number of segments one unit long that can be constructed on a geoboard of each size given and write the answers in the table.

Size of Geoboard	2 by 2	3 by 3	4 by 4	5 by 5	6 by 6	7 by 7	8 by 8	9 by 9	10 by 10
Number of Segments									

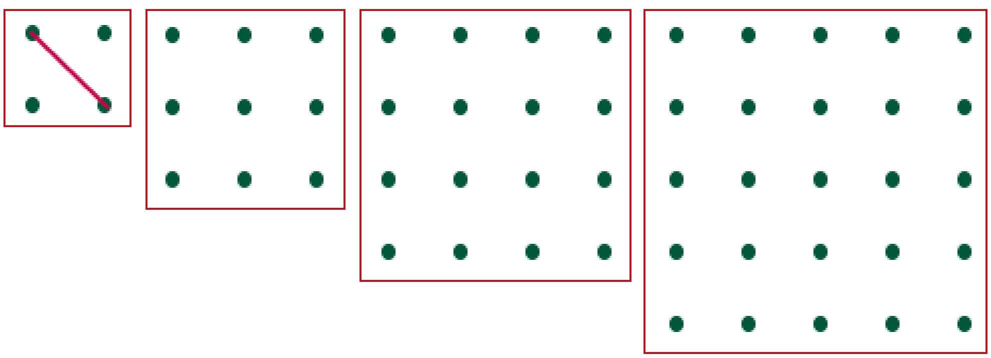
How many segments of this length could you place on an *n* by *n* geoboard? ________



Now make a segment that is as long as the diagonal of a 4-nail square. How many segments can be constructed with this length? Find the number of segments of this length that can be constructed on a geoboard of each size given and write the answers in the table.

Size of Geoboard	2 by 2	3 by 3	4 by 4	5 by 5	6 by 6	7 by 7	8 by 8	9 by 9	10 by 10
Number of Segments									

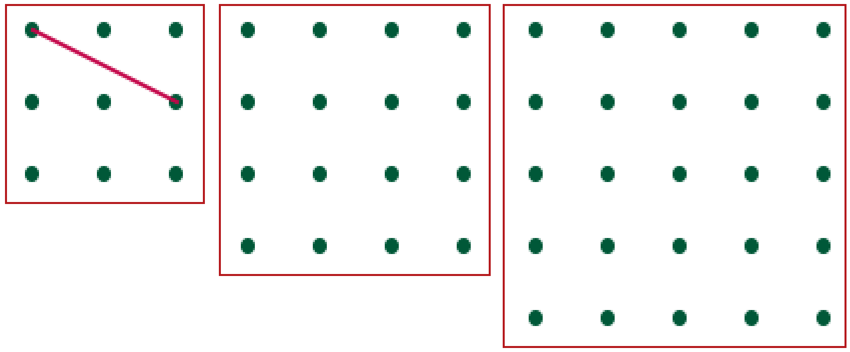
How many segments of this length can you place on an *n* by *n* geoboard?_________



Now make a segment that is as long as the diagonal of a 1 unit by 2 unit rectangle. How many segments can be constructed with this length? Find the number of segments of this length that can be constructed on a geoboard of each size given and write the answers in the table.

Size of Geoboard	2 by 2	3 by 3	4 by 4	5 by 5	6 by 6	7 by 7	8 by 8	9 by 9	10 by 10
Number of Segments									

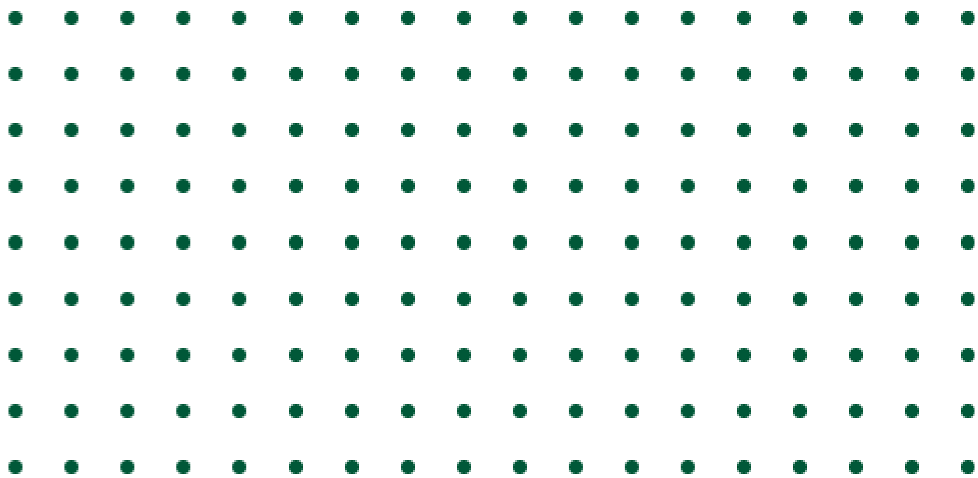
How many segments of this length can you place on an *n* by *n* geoboard?_________



This process can be continued for a segment of any given length on a geoboard. Use the dot paper below to investigate the solution to the following generalized problem situation:

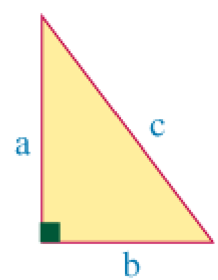
How many segments can be constructed on an *n* by *n* geoboard where each segment is the length of a diagonal of a rectangle *a* units wide by *b* units long?___________

Suppose that the rectangle is a square with *a* units on a side._________________

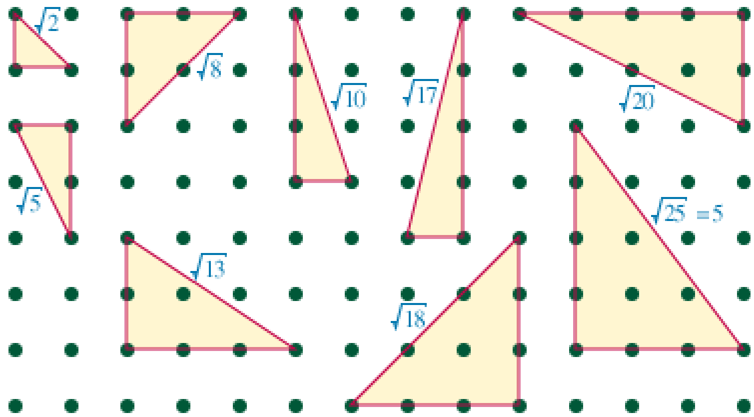


Length of a Segment

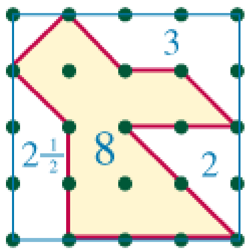
The length of any segment connecting two nails can be found by using the Pythagorean Theorem, which states that if the two legs of a right triangle have lengths of *a* and *b* respectively, and the hypotenuse has a length of *c*, then the following relationship is always true as is shown in the diagram at the right.



$$a^2 + b^2 = c^2$$



Area of a Figure



Rectangle has area 16.
Outside figure is 8.
Inside figure is 8.

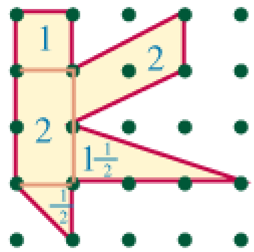
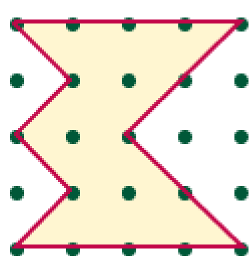
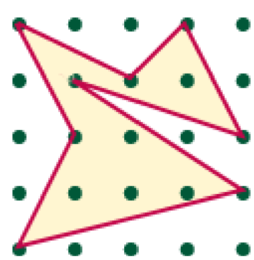
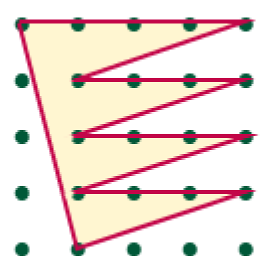


Figure is broken into smaller regions whose total area is 7

The area of a closed figure on the geoboard can be found by enclosing the figure with a rectangle, finding the area of the rectangle, and subtracting the areas of the portions not in the figure. A second approach to finding the area of a closed figure is to break the figure into smaller regions whose areas are easily determined.

Construct each of the figures shown below on a geoboard,using rubber bands to either enclose the figures or to separate the figures into smaller regions so that the area of each may be easily found. Draw lines on the figures to show where the rubber bands were placed. Find the length of each segment in each figure and add these lengths to find the perimeter of each figure. Write the area and perimeter below each figure.

Figure			
Area			
Perimeter			

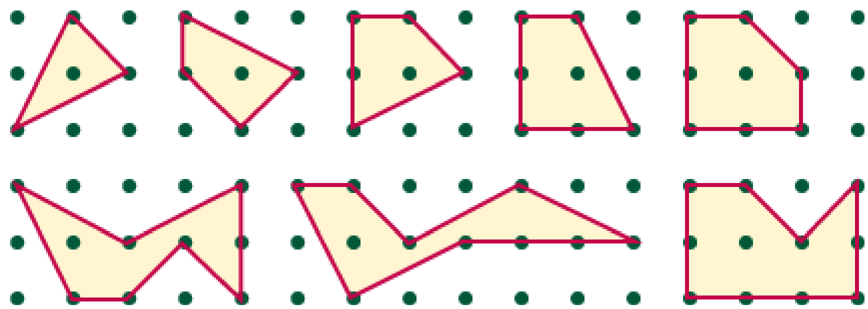
Pick's Formula for Area

It is possible to find the area of a figure made on a geoboard without using either of the techniques described

above. A formula known as Pick's Formula can be found by examining different categories of figures and finding their areas. It will be necessary to count the number of nails inside each figure but not touched by the figure, as well as the number of nails actually touched by the figure. We will first examine figures that have exactly one nail untouched inside the figure. Construct several figures of this type, count the number of nails touched, and find the area of each figure, recording your results in the table below. Use this information to try to guess the formula for *n* nails touched.

One Nail Inside Figure

Nails Touched	3	4	5	6	7	8	9	10	*n*
Area of Figure									



Examine some figures that have no nails untouched inside the figure, count the number of nails touched, and find the area of each figure, recording your results in the table below. Guess at any missing areas and try to guess the formula for *n* nails touched.

No Nails Inside Figure

Nails Touched	3	4	5	6	7	8	9	10	*n*
Area of Figure									

Examine some figures that have exactly two nails untouched inside the figure, count the number of nails touched, and find the area of each figure, recording your results in the table below. Guess at any missing areas and try to guess the formula for *n* nails touched.

Two Nails Inside Figure

Nails Touched	3	4	5	6	7	8	9	10	*n*
Area of Figure									

Examine some figures that have exactly three nails untouched inside the figure, count the number of nails touched, and find the area of each figure, recording your results in the table below. Guess at any missing areas and try to guess the formula for *n* nails touched.

Three Nails Inside Figure

Nails Touched	3	4	5	6	7	8	9	10	*n*
Area of Figure									

Use the results of these investigations to complete the following table with the formulas for the areas when *n* nails are touched. Generalize these formulas to the case when k nails are untouched inside the figure. This generalized formula gives the area of a figure that touches *n* nails with *k* nails untouched inside the figure and is called Pick's Formula.

k Nails Inside Figure

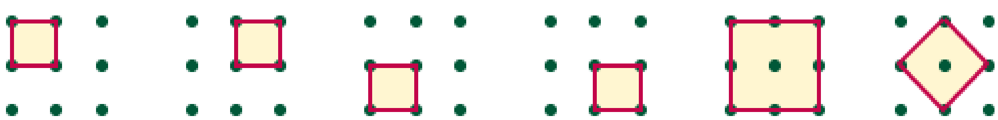
Nails Inside Figure	0	1	2	3	4	5	*k*
Formula for *n* Nails							

Squares on a Geoboard

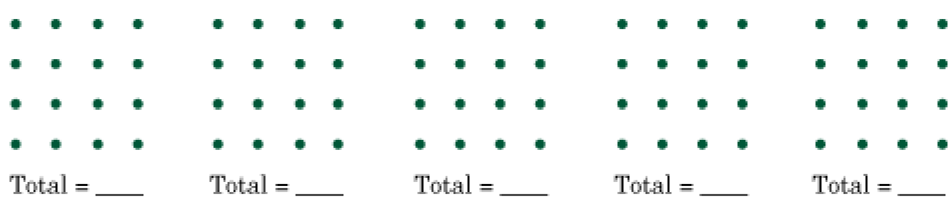
A fascinating problem on the geoboard is to determine how many squares can be constructed. It is necessary to consider all possible sizes and all possible positions. Squares may be made on a geoboard so that the base of the square either is or is not parallel to the base of the geoboard. To solve the problem in all its generality, we need to examine geoboards of different sizes.

A 2 by 2 geoboard can have only one square, and this square is parallel to the base of the geoboard.

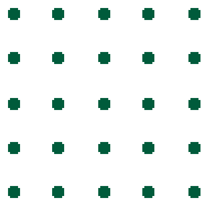
A 3 by 3 geoboard can have 6 squares in all. Five of these squares are parallel to the base of the geoboard. There are 3 different squares shown of the 6 possible. The squares are shown on the dot paper below.

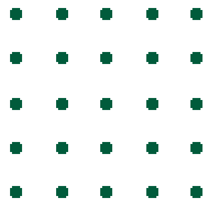


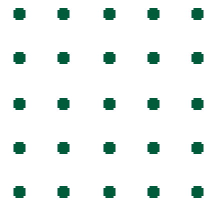
Find all the squares that can be drawn on a 4 by 4 geoboard. Begin by finding all the different squares that are parallel to the base and the number of each size. Next, find all the squares that are not parallel to the base and the number of each. Draw the different sizes of squares and record the number of each below the dot grid.



Find all the squares that can be drawn on a 5 by 5 geoboard. Begin by finding all the different squares that are parallel to the base and the number of each size. Next, find all the squares that are not parallel to the base and the number of each. Draw the different sizes of squares and record the number of each below the dot grid.


Total = _____


Total = _____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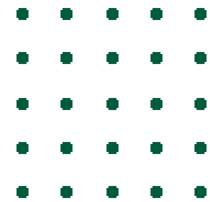
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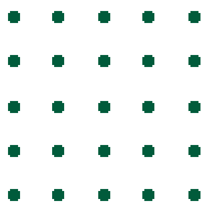
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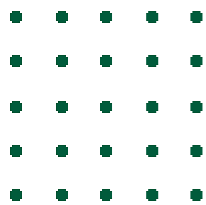
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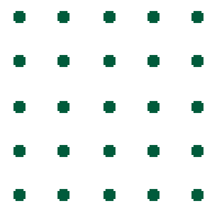
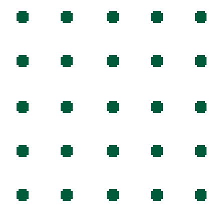

Total = _____


Total = _____

Use the results of your investigations to fill in the table below. Use the patterns in the table to extend the results to 6 by 6, 7 by 7, 8 by 8, 9 by 9, and 10 by 10 geoboards.

	Parallel to Base		Not Parallel to Base		
Size	**Different**	**Total**	**Different**	**Total**	**Grand Total**
1 by 1	0	0	0	0	0
2 by 2	1	1	0	0	1
3 by 3	2	5	1	1	6
4 by 4					
5 by 5					
6 by 6					
7 by 7					
8 by 8					
9 by 9					
10 by 10					

The results in the table above can be used to generalize to an n by n geoboard using the method of *Finite Differences*. Using this method, the values for each size are placed in a column next to a column of different sizes and the differences between consecutive values are computed. If the differences are not constant, the second differences are computed, and so on, until the differences are constant. The formula derived will be a polynomial in n whose degree will be the order of the differences calculated. The number of terms will be one more than the order. Therefore the formula will be of the form

$$a_1n^k + a_2n^{k\text{-}1} + a_3n^{k\text{-}2} + \ldots + a_kn + a_{k+1}$$

where k is the order of the differences and n is the size. The value of a_1 is equal to the constant difference divided by k!.

$$a_1 = \text{constant} / k! = \text{constant} / k(k\text{-}1)(k\text{-}2)\ldots(3)(2)(1)$$

The example below shows the formula for the total number of squares on an n by n geoboard that are parallel to the base.

Size	0	1	2	3	4	5	6	7	8	9	10
Squares	0	0	1	5	14	30	55	91	140	204	285
1st Diff		0	1	4	9	16	25	36	49	64	81
2nd Diff			1	3	5	7	9	11	13	15	17
3rd Diff				2	2	2	2	2	2	2	2

Since the third order differences are constant and the value of that constant is 2, then the formula becomes

$$^1/_3\, n^3 + bn^2 + cn + d$$

The value of *d* can be found by replacing *n* by 0 and the formula by its value when *n* is 0, namely 0. The first three terms are 0; so then, $d = 0$, and the formula now becomes

$$^1/_3\, n^3 + bn^2 + cn$$

The values of *b* and *c* can be found by taking two different values for *n* and replacing *n* by these values in the formula, and solving these two equations simultaneously.

Solving the two equations simultaneously, it is found that $b = {}^{-1}/_2$ and $c = {}^1/_6$, giving the final formula as

$$^1/_3\, n^3 - {}^1/_2\, n^2 + {}^1/_6\, n$$

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